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NEW METHOD FOR INVESTIGATING OF BIFURCATION REGIMES BY USE OF REALIZATIONS OF A DYNAMICAL SYSTEM

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In this paper we suggest a new characteristic of chaos which is the evolutionary order parameter. This parameter allows to determine bifurcation regimes by use of realization of a dynamical system. Unlike existing methods for constructing bifurcation diagrams, this method determines a state of the dynamical system more precisely. As a signal generator we select dynamical systems, such as logistic map, Henon map, the new bursting map, modified Lorenz system. We present bifurcation diagrams depending on evolutionary order parameter.

Keywords: Bifurcation, map, chaotic oscillations, evolutionary order parameter

Introduction

In modern research, bifurcations in different dynamical systems described as usual by use of the equation for cubic system with a degenerate saddle point [1], a singular nonlinear Sturm-Liouville equation [2], a reaction-diffusion equation with spatio-temporal delay [3], a model of Hydrogen-Bonded-Chains [4], the Hyperchaotic Oscillator with Gyrators [5] are considered. Bifurcation diagrams in these systems can be constructed as a dependence of maximum and minimum values of the physical quantity on control parameter given by equations for the description of the dynamical system.

However, equations of the dynamical system are not always known and so question arises whether it is possible to determine bifurcation regimes by realizations (time series, photographic images, etc.). Therefore, the aim of the paper is to determine the appropriate order parameter, which variation would lead to bifurcations.

For this purpose, we suggest a new expression for order parameter of an evolutionary process. This parameter allows to construct a bifurcation diagram by its realization without equations for the dynamical system.

1. Evolutionary order parameter of the highly heterogeneous chaotic signals

Existence of metric characteristics (length, area, volume) follows from execution of the well-known integral Holder inequality for all functions $x_i(t), x_j(t)$, written in the form

$$\left(\frac{1}{T}\int_0^T |x_i(t)|^p\, dt\right)^{1/p} \left(\frac{1}{T}\int_0^T |x_j(t)|^q\, dt\right)^{1/q} \le K_{x_i,x_j}^{p,q}\, \frac{1}{T}\int_0^T \left|x_i(t)\cdot x_j(t)\right| dt, \quad \frac{1}{p}+\frac{1}{q}=1, \tag{1}$$

where $K_{x_i,x_j}^{p,q}$ - is a coefficient. Constant value of the coefficient provides equality in Eq. (1). In Eq. (1) we use averaging in the time domain t. Denoting by the angle brackets more general averaging by ensemble, from (1) we have

$$K_{x_i,x_j}^{p,q} = \frac{\left(\left\langle \left|x_i\right|^p \right\rangle\right)^{1/p} \left(\left\langle \left|x_j\right|^q \right\rangle\right)^{1/q}}{\left|< x_i x_j >\right|}, \quad \frac{1}{p} + \frac{1}{q} = 1. \qquad (2)$$

Expression (2) is called the generalized - metric characteristics. This characteristic of chaos was introduced for the first time. Equation (2) is different from the reverse non-centered autocorrelation coefficient: modulus of the product of functions is averaged, the possibility of $p \neq q \neq 2$ is taken into account. In case of $p = q = 2$ the desired characteristic is determined by the Euclidean metric. If $x_{\mathrm{i}}(t) \equiv x(t)$, $x_j(t) = 1$, $p = q = 2$, we get $K_{x,1}^{2,2} = (\left\langle x^2 \right\rangle)^{1/2} / \left\langle |x| \right\rangle$ - form coefficient of a signal used in radiophysics.

Let us consider the possibility of using the formula (2) for chaotic signals which are strongly heterogeneous and asymmetrically intermittent. The intermittent functions are strongly inhomogeneous relatively to each other (x_i, x_j) and relatively to the argument (x,t).In terms of the similarity theory and scale invariance the intermitted signals do not have the property of self-similarity, and can be self-affine. To accommodate such nonequilibrium due to an arbitrary function $x_i(t)$, $x_j(t)$ in the formula (2) we can choose one of them as a determining variable. If we are interested in time evolution $x_i(t)$, we can choose $x_j(t) = t$.Then the expression (2) has the form

$$K_{x,t}^{p,q} = (\left\langle \left|x(t)\right|^p \right\rangle)^{1/p} \cdot (\left\langle t^q \right\rangle)^{1/q} / \left\langle \left|x(t) \cdot t\right| \right\rangle. \qquad (3)$$

Expression (3) is called the evolutionary order parameter. This parameter has meaning of dimensionless time and it is proportional to number of discrete steps of maps of a dynamical systems. If we accept $q = 2 + D_C$, $p = q/(q-1)$ in (2), then it is possible to increase resolution of the generalized - metric characteristics, because correlation dimension D_C is an important quantitative characteristics of the attractor, which carries information about the degree of behavior complexity of dynamical system. [6]. Algorithm for calculation of D_C is based on calculation of correlation integral $C(\delta)$ for normalized number of pairs of points of the object. Distance between these pairs of papers is not greater than δ:

$$C(\delta) = \frac{1}{N^2} \sum_{i \neq j} \theta(\delta - \left|x_i - x_j\right|), \qquad (4)$$

where $\theta(x)$ is Heaviside step function for all pairs of values of *i* and *j*. The value of sum depends on δ if this dependence has an exponential form

$$C(\delta) \approx \delta^{D_C}, \qquad (5)$$

In this case the investigated set can be considered as a fractal set in a defined range of δ.

2. Results of a numerical analysis and discussion

At first we show that the widely used characteristics of the signal dispersion and the base defined by the formulas

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \langle x\rangle)^2}, \qquad (6)$$

$$B = 2\tau \cdot \Delta w, \qquad (7)$$

where $\Delta w = \frac{1}{E_{\max}}\int_0^{\infty} E(w)dw,\ \tau = \frac{1}{R(0)}\int_0^{\infty} R(\tau)\,d\tau,\ R(\tau) = \int_0^{\infty} x(t-\tau)\cdot x(t)dt,$ are less informative than the evolutionary order parameter. This conclusion we have tested on different chaotic signals. We chose the logistic, one-dimensional map [6]

$$x_{i+1} = rx_i(1-x_i), \qquad (8)$$

where *r* is control parameter, two-dimensional map of Henon [7],

$$x_{i+1} = 1 - ax_i^2 - by_i,\ \ y_{i+1} = x_i,. \qquad (9)$$

where *a* and *b* are control parameters.

We propose a new map, which is deduced with use of limitation conditions of fractal measure derivative:

$$x_{i+1} = (\frac{1}{c} + \mu_i)\left|x_i\right|^{-\frac{1}{\gamma}}, \qquad \mu_{i+1} = -\frac{1}{\gamma}(\frac{1}{c} + \mu_i)\left|x_i\right|^{-\frac{1}{\gamma}-1}, \qquad (10)$$

where γ is the fractional part of the fractal dimension of the set of values of the considered physical quantity, c is the parameter, which is analogue of fractal signal base, reverse value of which determines accuracy of observation, μ_i is a multiplier. Map (10) can be considered as a bursting map and describes a chaotic alternation of small-scale and large-scale oscillations such as bursting, that clearly illustrates the effectiveness of using of the parameter $K_{x,t}^{p,q}$. For the purpose of application of this method to the analysis of a new type of "gluing" bifurcation we have also used a modified Lorenz system [8,9]

$$\frac{dy}{dt} = R\cdot x - y - x\cdot z, \quad \frac{dx}{dt} = \sigma\cdot(y-x) + A\cdot y\cdot z, \quad \frac{dz}{dt} = x\cdot y - b\cdot z, \qquad (11)$$

where σ, A, R, b are parameters of the system.

Results of signal processing characterizing the above mentioned systems are shown in Figures 1 and 2. Dispersion and signal base almost don't change at $K_{x,t}^{p,2+D_c} \geq 1.1$, i.e. evolutionary parameter is more sensitive. Bifurcation diagram constructed by the formula (8) by the standard method is shown in Fig. 3. By changing a parameter *r* of Feigenbaum map corresponding system changes its regimes of evolution with a period-doubling cascade which leads to chaos.

By using realizations of the formula (8) for different values of *r* we have constructed the bifurcation diagram via the evolutionary order parameter defined by the formula (3) at $p = q = 2$. As a result, we have got the bifurcation diagram, which shows a doubling and tripling of the period (Fig. 4). In order to increase resolution of the bifurcation analysis, we have used the formula (3) with the value $q = 2 + D_c$ and re-constructed the bifurcation diagram (Fig. 5).

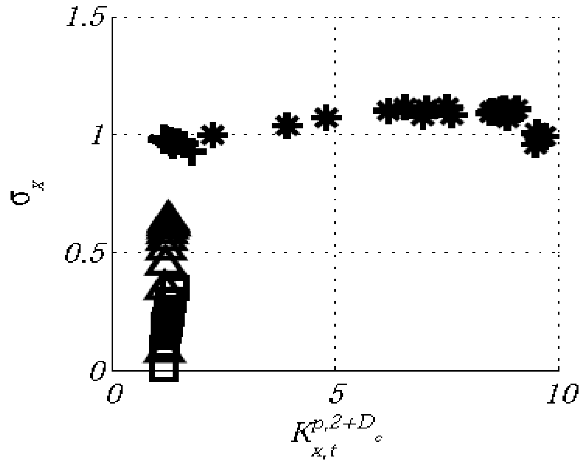

Fig. 1. Interdependence of dispersion with the evolutionary order parameter. (□) – logistic map at r =[3:0.1:4], (Δ) – Henon map at *b* = 0:1; *a* = [0.61:0.05:1.3], (*) - bursting map at c = 2:806, *γ*=[1:0.01:2], (+)- modified Lorenz system at *R* = 3; *b* = 8/3; *σ*=10, *A*=[10.5:0.5:16]

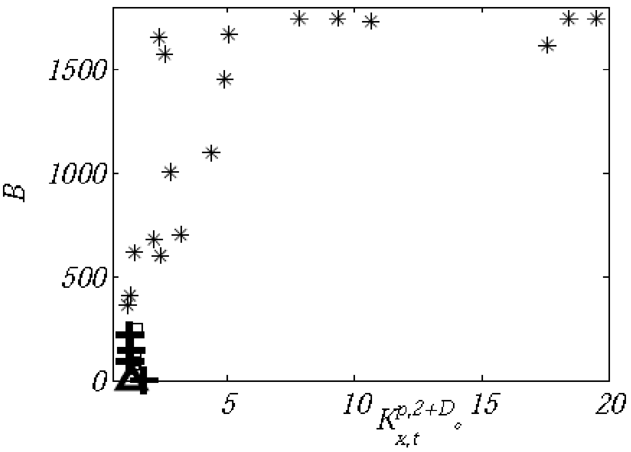

Fig. 2. Dependence of signal base on the evolutionary order parameter. (□) – logistic map at r =[3:0.1:4], (Δ) – Henon map at *b* = 0:1; *a* = [0.61:0.05:1.3], (*) - bursting map at c = 2:806, *γ*=[1:0.01:2], (+)- modified Lorenz system at *R* = 3; *b* = 8/3; *σ*=10, *A*=[10.5:0.5:16]

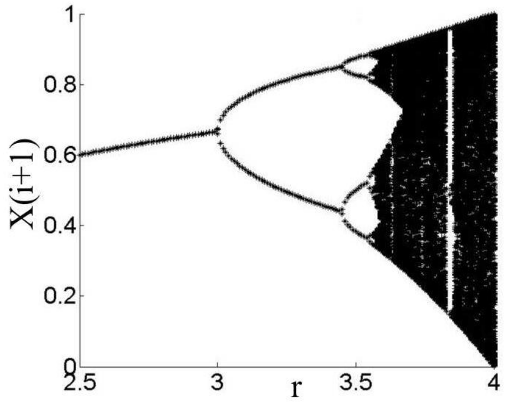

Fig. 3. The bifurcation diagram of the logistic map (8) by the standard method

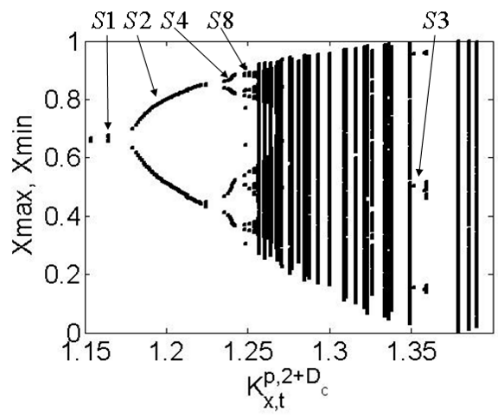

Fig. 4. The bifurcation diagram of the logistic map which constructed by the realization of the formula (8) with *r* = 2:5:0:001:4, by using a new characteristics $K_{x,t}^{2,2}$

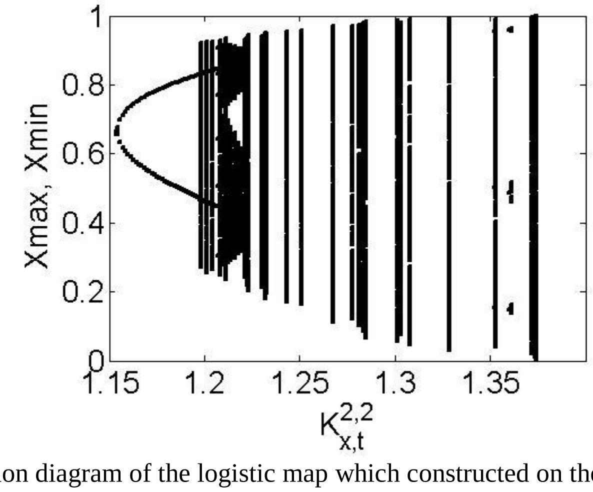

Fig. 5. The bifurcation diagram of the logistic map which constructed on the evolutionary order parameter at *q*=2+D_C; *p*=*q*/(*q*-1); *r*=2.5:0.001:4

We have shown all cycles in Fig. 5: S1 (limit cycle), S2 (period doubling), S3 (period tripling) and transition to chaos. Thus, all periodic realizations are collected on the left side, and they are chaotic on the right side of the bifurcation diagram. So, we have a possibility to classify different bifurcations in dynamical systems with unknown parameters.

Variation of parameters a; b in the two-dimensional system (9) may lead to multistability. Multistability can be considered as coexistence of two or more different dynamical regimes, such as chaotic attractor and cycle of period n or 2 different in structure chaotic sets.

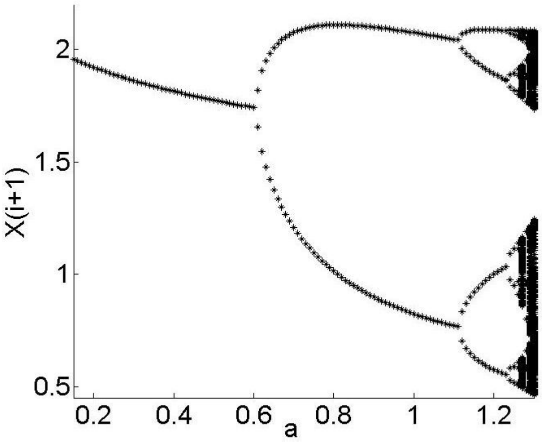

Fig. 6. Bifurcation diagram of the Henon map at a=0.15:0.01:1.3, b=0.1

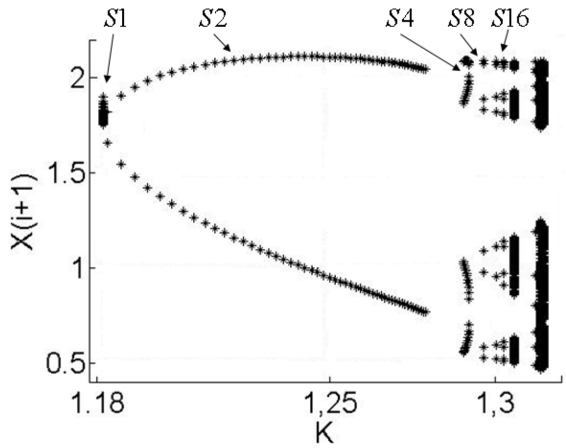

Fig. 7. Bifurcation diagram of the Henon map vs. evolutionary order parameter at a=0.61:0.001:1.3, b=0.1, $K \equiv K_{x,t}^{p,q+D_c}$

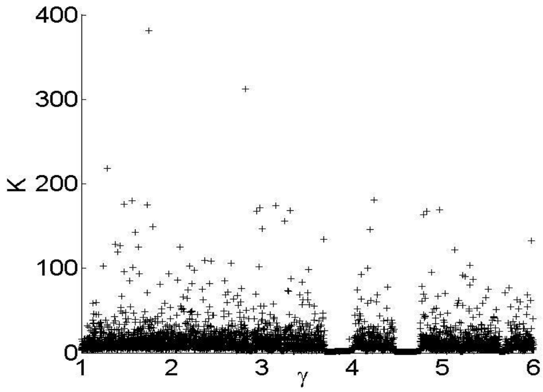

Fig. 8. Dependence of the evolutionary order parameter on γ of the bursting map at c=2,806, $K \equiv K_{x,t}^{p,q+D_c}$

Bifurcation diagrams of the two-dimensional map constructed by the known parameter in equation (9) (Fig. 6) and by the new method (Fig. 7) are also different. It is clearly seen the advantage of our proposed method. At some values of $K_{x,t}^{p,q}$ there are missing values of X(i+1). This means that some cycles of a possible bifurcation sets *S1*, *S2*, *S3* and their formations are missing.

Values of $K_{x,t}^{p,q}$ are missing in some bands, because dependence of $K_{x,t}^{p,q}(\gamma)$ is discontinuous (Fig. 8).

Map (10) realizes an asymmetric alternation with strong bursts on the background of small-scale oscillations i.e. signals such as bursting (Fig. 9).

Values of the parameter $K_{x,t}^{p,q}$ for periodic oscillations are minimal, but for oscillations with explosive character values of $K_{x,t}^{p,q}$ are maximal.

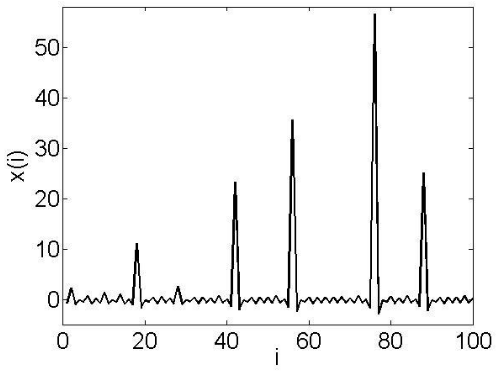

Fig. 9. Realization of bursting map (formula (10)) at c=2,806, γ=3.33

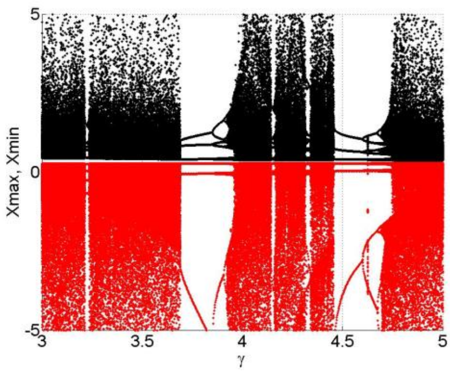

Fig. 10. Bifurcation diagram of the bursting map at c=2,806.

At small values of the parameter $K_{x,t}^{p,q}$ we see a typical period doubling, i.e. Feigenbaum transition scenarios leading to appearance of chaos (Fig. 10).

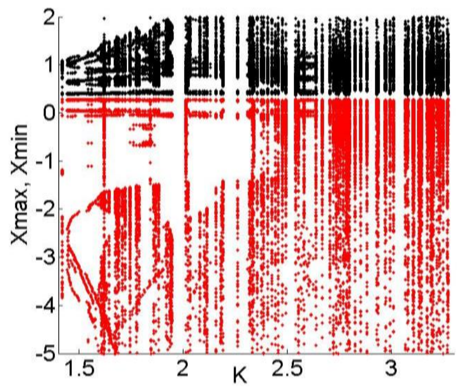

Fig. 11. Bifurcation diagram of the bursting map use the evolutionary order parameter at $q=2+D_C$; $p=q/(q-1)$; c=2,806, γ=3.5:0.01:5, $K \equiv K_{x,t}^{p,q}$

Bifurcation diagram constructed with considering a correlation dimension (via $K_{x,t}^{p,q+D_c}$) is more detailed (Fig. 11). All stable regimes are localized in the left side of the bifurcation diagram. High values of $K_{x,t}^{p,q+D_c}$ correspond to chaotic structure. So bifurcation pattern becomes more ordered: qualitatively different regimes are grouped. We see pattern of Feigenbaums period-doubling (cycle S2) in different intervals of evolutionary parameter. Some branches of the inclined lines of doubling bifurcations are not realized, the process has an asymmetry. Changing the parameter c at a constant value γ, we get a similar bifurcation pattern.

We applied our method to the investigation of a special type of homoclinic bifurcation (gluing bifurcation). For this purpose, we use a system of differential equations (11). Choosing A as the control parameter, bifurcation diagram was constructed by the standard method (Fig. 12). We see that there are points of "gluing", which are also stable points near x = 0. Thus, the upper and lower bifurcation diagrams correspond to the Feigenbaum scenario transition.

Bifurcation diagram of the system (12) plotted by use of our method is shown in Fig. 13. The main difference between Fig. 12 and Fig. 13 is that cycles 1, 2, 4, 8, 16, 32, ... and asymmetric cycles 2a, 4a, 8a, 16a, ... are grouped. Cycle 1 is a loop in the phase space. Asymmetrical cycle 2a is realized after cycles 1, 2, 4, 8, and cycles 4a, 8a, 16a are realized after cycles 16, 32. That means that asymmetrical cycles are more complex than symmetric.

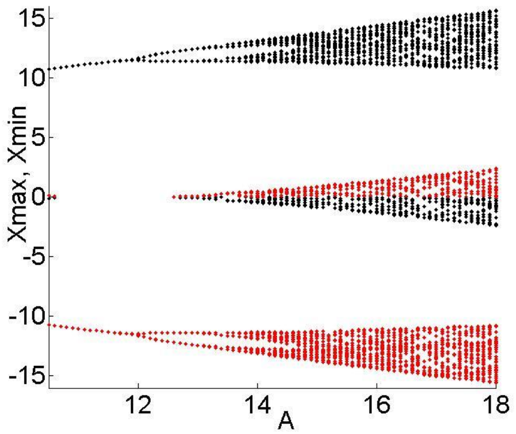

Fig. 12. Homoclinic bifurcation of a system of differential equations (11) at $\sigma = 10$, $b = 2.67$, $R = 3$ *A*= 10,5:0,01:18

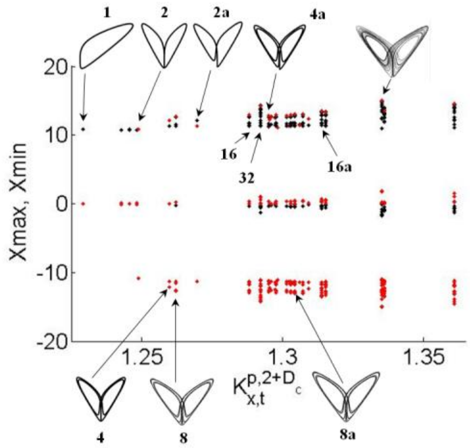

Fig. 13. Homoclinic bifurcation of a system of differential equations (11) use the evolutionary order parameter at $q=2+D_C$; $p=q/(q-1)$; $\sigma = 10$, $b = 2.67$, $R = 3$, *A*=10,5:0,01:18

Conclusion

In the present work we suggest a new evolutionary order parameter. We constructed bifurcation diagrams of dynamical systems: (logistic map, Henon map, map for "bursting" type oscillations and for systems with homoclinic bifurcations) via this parameter.

We compared these diagrams with the diagrams constructed by standard methods via the parameters of the dynamical system equations. The main advantage of our method is that it gives us a possibility to investigate the bifurcation phenomena by realizations without knowing initial equations. Our new method allows us to get more detailed patterns of bifurcations and automatically groups cycles. This method can be used for analysis of various complex phenomena.

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